Strong insertion of a contra-Baire-1 (Baire-.5) function

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Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

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1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [18]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [18].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [25] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuity. A considerable number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 8, 9, 10, 12, 13, 23].

The results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

A real-valued function f defined on a topological space X is called *contra-Baire-1 (Baire-.5)* if the preimage of every open subset of $\mathbb{R}$ is a $G_\delta-$set in X [26].

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If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of the conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a $B-.5-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P. If P_1 and P_2 are $B-.5-$properties, the following terminology is used: (i) A space X has the *weak* $B-.5-$*insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$. (ii) A space X has the *strong* $B-.5-$*insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space that $F_\sigma-$kernel of sets are $F_\sigma-$sets, is given a sufficient condition for the weak $B-.5-$insertion property. Also, for a space with the weak $B-.5-$insertion property, we give necessary and sufficient conditions for the space to have the strong $B-.5-$insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the strong insertion of a contra-continuous function between two comparable real-valued functions has also recently considered by the authors in [21].

2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V, as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
In [7, 19, 22], A^Λ is called the *kernel* of A.

We also define the subsets $G_\delta(A)$ and $F_\sigma(A)$, as follows:
$G_\delta(A) = \cup\{O : O \subseteq A, O is G_\delta - set\}$ and
$F_\sigma(A) = \cap\{F : F \supseteq A, F is F_\sigma - set\}$.

$F_\sigma(A)$ is called the $F_\sigma - kernel$ of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ \nu$ implies $x \ \rho \ \nu$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

3) If $A\ \rho\ B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, that $F_\sigma-$kernel sets in X are $F_\sigma-$sets, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis, there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define the functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [15], it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is a $G_\delta-$set in X, i.e., h is a Baire-.5 function on X. □

The above proof used the technique of Theorem 1 of [14].

If a space has the strong $B - .5$-insertion property for (P_1, P_2), then it has the weak $B - .5$-insertion property for (P_1, P_2).The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $B - .5$-insertion property to satisfy the strong $B - .5$-insertion property.

Theorem 2.2. Let P_1 and P_2 be $B - .5-$property and X be a space that satisfies the weak $B - .5-$insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g \leq f, g$ has property P_1 and f has property P_2. The space X has the strong $B - .5-$insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f-g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n, F_n and $A(f-g, 2^{-n})$ are completely separated by Baire-.5 functions, and (ii)$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Suppose that there is a sequence $(A(f-g, 2^{-n}))$ of lower cut sets for $f-g$ and suppose that there is a sequence (F_n) of subsets of X such that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n, there exists a Baire-.5 function k_n on X into $[0, 2^{-n}]$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f-g, 2^{-n})$. The function k from X into $[0, 1/4]$ which is defined by

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x)$$

is a Baire-.5 function by the Cauchy condition and the properties of Baire-.5 functions, (1) $k^{-1}(0) = \{x \in X : (f-g)(x) = 0\}$ and (2) if $(f-g)(x) > 0$ then $k(x) < (f-g)(x)$: In order to verify (1), observe that if $(f-g)(x) = 0$, then $x \in A(f-g, 2^{-n})$ for each n and hence $k_n(x) = 0$ for each n. Thus $k(x) = 0$. Conversely, if $(f-g)(x) > 0$, then there exists an n such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f-g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f-g, 2^{-n})$$

and that $(A(f-g, 2^{-n}))$ is a decreasing sequence. Thus if $(f-g)(x) > 0$ then either $x \notin A(f-g, 1/2)$ or there exists a smallest n such that $x \notin A(f-g, 2^{-n})$ and $x \in A(f-g, 2^{-j})$ for $j = 1, \ldots, n-1$.
In the former case,

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f-g)(x),$$

and in the latter,

$$k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \leq 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f-g)(x).$$

Thus $0 \leq k \leq f-g$ and if $(f-g)(x) > 0$ then $(f-g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$ then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are $B-.5-$properties, then g_1 has property P_1 and f_1 has property P_2. Since by hypothesis X has the weak $B-.5-$insertion property for (P_1, P_2) , then there exists a Baire-.5 function h such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Therefore X has the strong $B-.5-$insertion property for (P_1, P_2). (The technique of this proof is by Lane [16].)

Conversely, assume that X satisfies the strong $B-.5-$insertion for (P_1, P_2). Let g and f be functions on X satisfying P_1 and P_2 respectively such that $g \leq f$. Thus there exists a Baire-.5 function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. We follow an idea contained in Powderly [24]. Now consider the functions 0 and $f-h.0$ satisfies property P_1 and $f-h$ satisfies property P_2. Thus there exists a Baire-.5 function h_1 such that $0 \leq h_1 \leq f-h$ and if $0 < (f-h)(x)$ for any x in X, then $0 < h_1(x) < (f-h)(x)$. We next show that

$$\{x \in X : (f-g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.$$

If x is such that $(f-g)(x) > 0$, then $g(x) < f(x)$. Therefore $g(x) < h(x) < f(x)$. Thus $f(x) - h(x) > 0$ or $(f-h)(x) > 0$. Hence $h_1(x) > 0$. On the other hand, if $h_1(x) > 0$, then since $(f-h) \geq h_1$ and $f-g \geq f-h$, therefore $(f-g)(x) > 0$. For each n, let

$$A(f-g, 2^{-n}) = \{x \in X : (f-g)(x) \leq 2^{-n}\}, \quad F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}$$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Since $\{x \in X : (f-g)(x) > 0\} = \{x \in X : h_1(x) > 0\}$, it follows that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is a Baire-.5 function which completely separates F_n and $A(f-g, 2^{-n})$. From its definition and by the properties of Baire-.5 functions, it is clear that k_n is a Baire-.5 function. Let $x \in F_n$. Then, from the definition of $k_n, k_n(x) = 2^{-n}$. If $x \in A(f-g, 2^{-n})$, then since $h_1 \leq f-h \leq f-g, h_1(x) \leq 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of k_n. Hence k_n completely separates F_n and $A(f-g, 2^{-n})$. □

Theorem 2.3. Let P_1 and P_2 be $B-.5-$properties and assume that the space X satisfied the weak $B-.5-$insertion property for (P_1, P_2). The space X satisfies the strong $B-.5-$insertion property for (P_1, P_2) if and only if X satisfies the strong $B-.5-$insertion property for $(P_1, B-.5)$ and for $(B-.5, P_2)$.

Proof. Assume that X satisfies the strong $B-.5-$insertion property for $(P_1, B-.5)$ and for $(B-.5, P_2)$. If g and f are functions on X such that $g \leq f, g$ satisfies property P_1, and f satisfies property P_2, then since X satisfies the weak $B-.5-$insertion property for (P_1, P_2) there is a Baire-.5 function k such that $g \leq k \leq f$. Also, by hypothesis there exist Baire-.5 functions h_1 and h_2 such that $g \leq h_1 \leq k$ and if $g(x) < k(x)$ then $g(x) < h_1(x) < k(x)$ and such that $k \leq h_2 \leq f$ and if $k(x) < f(x)$ then $k(x) < h_2(x) < f(x)$. If a function h is defined by $h(x) = (h_2(x) + h_1(x))/2$, then h is a Baire-.5 function, $g \leq h \leq f$, and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Hence X satisfies the strong $B-.5-$insertion property for (P_1, P_2). The converse is obvious since any Baire-.5 function must satisfy both properties P_1 and P_2. (The technique of this proof is by Lane [17].) □

3 Applications

Definition 3.1. A real-valued function f defined on a space X is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a $G_\delta-$set for any real number t.

The abbreviations $usc, lsc, cusB_.5$ and $clsB_.5$ are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1. [14, 15]. A space X has the weak $c-$insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of theorem 2.1, 2.2 and 2.3 we suppose that X is a topological space that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

Corollary 3.1. For each pair of disjoint $F_\sigma-$sets F_1, F_2, there are two $G_\delta-$sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ if and only if X has the weak $B-.5-$insertion property for $(cusB-.5, clsB-.5)$.

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1, g is usB_1, and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $F_\sigma-$set and since $\{x \in X : g(x) < t_2\}$ is a $G_\delta-$set, it follows that $F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2. 1.

On the other hand, let F_1 and F_2 are disjoint $F_\sigma-$sets. Set $f = \chi_{F_1^c}$ and $g = \chi_{F_2}$, then f is $clsB-.5, g$ is $cusB-.5$, and $g \leq f$. Thus there exists Baire-.5 function h such that $g \leq h \leq f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then G_1 and G_2 are disjoint $G_\delta-$sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. □

Remark 2. [27]. A space X has the weak $c-$insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. For every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set if and only if X has the weak $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For every G of $G_\delta-$set we have $F_\sigma(G)$ is a $G_\delta-$set.

(ii) For each pair of disjoint $G_\delta-$sets as G_1 and G_2 we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \varnothing$.

The proof of Lemma 3.1 is a direct consequence of the definition $F_\sigma-$kernel of sets.

We now give the proof of Corollary 3.2.

Proof. Let g and f be real-valued functions defined on the X, such that f is $clsB-.5, g$ is $cusB-.5$, and $f \leq g$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $F_\sigma(A) \subseteq$

$G \subseteq F_\sigma(G) \subseteq G_\delta(B)$ for some $G_\delta-$set g in X, then by hypothesis and Lemma 3.1 ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(g,t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\};$$
$$= A(f,t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a $G_\delta-$set and since $\{x \in X : f(x) \leq t_2\}$ is a $F_\sigma-$set, by hypothesis it follows that $A(g,t_1)\ \rho\ A(f,t_2)$. The proof follows from Theorem 2.1.

On the other hand, Let G_1 and G_2 are disjoint $G_\delta-$sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1^c}$, then f is $clsB - .5, g$ is $cusB - .5$, and $f \leq g$.

Thus there exists Baire-.5 function h such that $f \leq h \leq g$. Set $F_1 = \{x \in X : h(x) \leq \frac{1}{3}\}$ and $F_2 = \{x \in X : h(x) \geq 2/3\}$ then F_1 and F_2 are disjoint $F_\sigma-$sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence $F_\sigma(F_1) \cap F_\sigma(F_2) = \varnothing$. □

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.2. The following conditions on the space X are equivalent:

(i) Every two disjoint $F_\sigma-$sets of X can be separated by $G_\delta-$sets of X.

(ii) If F is a $F_\sigma-$set of X which is contained in a $G_\delta-$set G, then there exists a $G_\delta-$set H such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

Proof. (i) ⇒ (ii) Suppose that $F \subseteq G$, where F and G are $F_\sigma-$set and $G_\delta-$set of X, respectively. Hence, G^c is a $F_\sigma-$set and $F \cap G^c = \varnothing$.

By (i) there exists two disjoint $G_\delta-$sets G_1, G_2 such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \varnothing \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a $F_\sigma-$set containing G_1 we conclude that $F_\sigma(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) ⇒ (i) Suppose that F_1, F_2 are two disjoint $F_\sigma-$sets of X.

This implies that $F_1 \subseteq F_2^c$ and F_2^c is a $G_\delta-$set. Hence by (ii) there exists a $G_\delta-$set H such that, $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$.

But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \varnothing$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore, $(F_\sigma(H))^c$ is a $G_\delta-$set of X. Hence $F_1 \subseteq H, F_2 \subseteq (F_\sigma(H))^c$ and $H \cap (F_\sigma(H))^c = \varnothing$. This means that condition (i) holds. □

Lemma 3.3. Suppose that X is the topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. If F_1 and F_2 are two disjoint $F_\sigma-$sets of X, then there exists a Baire-.5 function $h : X \to [0,1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.

Proof. Suppose F_1 and F_2 are two disjoint $F_\sigma-$sets of X. Since $F_1 \cap F_2 = \varnothing$, hence $F_1 \subseteq F_2^c$. In particular, since F_2^c is a $G_\delta-$set of X containing F_1, by Lemma 3.2, there exists a $G_\delta-$set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that $H_{1/2}$ is a $G_\delta-$set and contains F_1, and F_2^c is a $G_\delta-$set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 3.2, there exists $G_\delta-$sets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $G_\delta-$sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that h is a Baire-.5 function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are $G_\delta-$sets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(F_\sigma(H_t))^c : t > \alpha\}$ hence, every of them is a $G_\delta-$set. Consequently h is a Baire-.5 function. □

Lemma 3.4. Suppose that X is the topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. If F_1 and F_2 are two disjoint $F_\sigma-$sets of X and F_1 is a countable intersection of $G_\delta-$sets, then there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h(F_2) = \{1\}$.

Proof. Suppose that $F_1 = \bigcap_{n=1}^\infty G_n$, where G_n is a $G_\delta-$set of X. We can suppose that $G_n \cap F_2 = \varnothing$, otherwise we can substitute G_n by $G_n \setminus F_2$. By Lemma 3.3, for every $n \in \mathbb{N}$, there exists a Baire-.5 function h_n on X into $[0,1]$ such that $h_n(F_1) = \{0\}$ and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^\infty 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is a Baire-.5 function from X to $[0,1]$. Since for every $n \in \mathbb{N}, F_2 \subseteq X \setminus G_n$, therefore $h_n(F_2) = \{1\}$ and consequently $h(F_2) = \{1\}$. Since $h_n(F_1) = \{0\}$, hence $h(F_1) = \{0\}$. It suffices to show that if $x \notin F_1$, then $h(x) \neq 0$.

Now if $x \notin F_1$, since $F_1 = \bigcap_{n=1}^\infty G_n$, therefore there exists $n_0 \in \mathbb{N}$ such that $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$. □

Lemma 3.5. Suppose that X is the topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. The following conditions are equivalent:

(i) For every two disjoint $F_\sigma-$sets F_1 and F_2, there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) Every $F_\sigma-$set is a countable intersection of $G_\delta-$set.
(iii) Every $G_\delta-$set is a countable union of $F_\sigma-$set.

Proof. (i) $\Rightarrow$ (ii). Suppose that F is a $F_\sigma-$sets. Since $\varnothing$ is a $F_\sigma-$set, by (i) there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F$. Set $G_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, G_n is a $G_\delta-$set and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) $\Rightarrow$ (i). Suppose that F_1 and F_2 are two disjoint $F_\sigma-$sets. By Lemma 3.4, there exists a Baire-.5 function f on X into $[0,1]$ such that $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two $F_\sigma-$sets and $(G \cup F) \cap F_2 = \varnothing$. By Lemma 3.4, there exists a Baire-.5 function g on X into $[\frac{1}{2}, 1]$ such that $g^{-1}(1) = F_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F.h$ is well-defined and a Baire-.5 function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0,1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every $F_\sigma-$set is a $G_\delta-$set and complement of every $G_\delta-$set is a $F_\sigma-$set, the equivalence is hold. $\square$

Remark 3. [20] . A space X has the strong $c-$insertion property for (usc, lsc) if and only if X is perfectly normal.

Corollary 3.3. For every two disjoint $F_\sigma-$sets F_1 and F_2, there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ if and only if X has the strong $B-.5-$insertion property for $(cusB-.5, clsB-.5)$.

Proof. Since for every two disjoint $F_\sigma-$sets F_1 and F_2, there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then G_1 and G_2 are two disjoint $G_\delta-$sets that contain F_1 and F_2, respectively. This means that, we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. Hence by Corollary 3.1, X has the weak $B-.5-$insertion property for $(cusB-.5, clsB-.5)$. Now, assume that g and f are functions on X such that $g \leq f, g$ is $cusB-.5$ and f is $clsB-.5$. Since $f-g$ is $clsB-.5$, therefore the lower cut set $A(f-g, 2^{-n}) = \{x \in X : (f-g)(x) \leq 2^{-n}\}$ is a $F_\sigma-$set. By Lemma 3.5, we can choose a sequence $\{F_n\}$ of $F_\sigma-$sets such that $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}, F_n$ and $A(f-g, 2^{-n})$ are disjoint. By Lemma 3.3, F_n and $A(f-g, 2^{-n})$ can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, X has the strong $B-.5-$insertion property for $(cusB-.5, clsB-.5)$.

On the other hand, suppose that F_1 and F_2 are two disjoint $F_\sigma-$sets. Since $F_1 \cap F_2 = \varnothing$, hence $F_2 \subseteq F_1^c$. Set $g = \chi_{F_2}$ and $f = \chi_{F_1^c}$. Then f is $clsB-.5$ and g is $cusB-.5$ and furthermore $g \leq f$. By hypothesis, there exists a Baire-.5 function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$ we have $g(x) < h(x) < f(x)$. By definitions of f and g, we have $h^{-1}(1) = F_2 \cap F_1^c = F_2$ and $h^{-1}(0) = F_1 \cap F_2^c = F_1$. $\square$

Remark 4. [2]. A space X has the strong $c-$insertion property for (lsc, usc) if and only if each open subset of X is closed.

Corollary 3.4. Every $G_\delta-$set is a $F_\sigma-$set if and only if X has the strong $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.

Proof. By hypothesis, for every G of $G_\delta-$set, we have $F_\sigma(G) = G$ is a $G_\delta-$set. Hence by Corollary 3.2, X has the weak $B-.5-$insertion property for $(clsB-.5, cusB-.5)$. Now, assume that g and f are functions on X such that $g \leq f, g$ is $clsB-.5$ and f is $B-.5$. Set $A(f-g, 2^{-n}) = \{x \in X : (f-g)(x) < 2^{-n}\}$. Then, since $f-g$ is $cusB-.5$, we can say that $A(f-g, 2^{-n})$ is a $G_\delta-$set. By hypothesis, $A(f-g, 2^{-n})$ is a $F_\sigma-$set. Set $G_n = X \setminus A(f-g, 2^{-n})$. Then G_n is a $G_\delta-$set. This means that G_n and $A(f-g, 2^{-n})$ are disjoint $G_\delta-$sets and also are two disjoint $F_\sigma-$sets. Therefore G_n and $A(f-g, 2^{-n})$ can be completely separated by Baire-.5 functions. Now, we have $\bigcup_{n=1}^{\infty} G_n = \{x \in X : (f-g)(x) > 0\}$. By Theorem 2.2, X has the strong $B-.5-$insertion property for $(clsB-.5, B-.5)$. By an analogous argument, we can prove that X has the strong $B-.5-$insertion property for $(B-.5, cusB-.5)$. Hence, by Theorem 2.3, X has the strong $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.

On the other hand, suppose that X has the strong $B-.5-$insertion property for $(clsB-.5, cusB-.5)$. Also, suppose that G is a $G_\delta-$set. Set $f = 1$ and $g = \chi_G$. Then f is $cusB-.5, g$ is $clsB-.5$ and $g \leq f$. By hypothesis, there exists a Baire-.5 function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$, we have $g(x) < h(x) < f(x)$. It is clear that $h(G) = \{1\}$ and for $x \in X \setminus G$ we have $0 < h(x) < 1$. Since h is a Baire-.5 function, therefore $\{x \in X : h(x) \geq 1\} = G$ is a $F_\sigma-$set, i.e., G is a $F_\sigma-$set. □

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